

Twice Q -polynomial distance-regular graphs of diameter 4

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Abstract

It is known that a distance-regular graph with valency k at least three admits at most two Q -polynomial structures. In this note we show that all distance-regular graphs with diameter four and valency at least three admitting two Q -polynomial structures are either dual bipartite or almost dual imprimitive. By the work of Dickie [5] this implies that any distance-regular graph with diameter d at least four and valency at least three admitting two Q -polynomial structures is, provided it is not a Hadamard graph, either the cube $H(d, 2)$ with d even, the half cube $1/2H(2d+1, 2)$, the folded cube $\tilde{H}(2d+1, 2)$, or the dual polar graph on $[^2A_{2d-1}(q)]$ with $q \geq 2$ a prime power.

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1. Introduction

Dickie [5] showed that any distance-regular graph with diameter at least four and valency at least three admits at most two Q -polynomial structures, extending the work of Bannai and Ito who showed it for diameter at least 34. For brevity we call a distance-regular graph (or an association scheme) with exactly two Q -polynomial structures *twice Q -polynomial*. Furthermore, Dickie [5] classified twice Q -polynomial distance-regular graphs with diameter at least five and valency at least three.

Theorem 1. [5, Theorem 8.1.2] *Let Γ be a distance regular graph with diameter $d \geq 5$ and valency $k \geq 3$. Then Γ has two Q -polynomial structures if and only if Γ is one of the following:*

- (i) *the cube $H(d, 2)$ with d even;*
- (ii) *the half cube $\frac{1}{2}H(2d+1, 2)$;*
- (iii) *the folded cube $\tilde{H}(2d+1, 2)$;*
- (iv) *the dual polar graph on $[^2A_{2d-1}(q)]$, where $q \geq 2$ is a prime power.*

In this note we show that Theorem 1 can be extended to include the diameter four case. The following result is key to doing so.

Theorem 2. *Let Γ denote a twice Q -polynomial distance-regular graph of diameter four and valency at least three. Then one of the Q -polynomial structures has $a_1^* = a_2^* = a_3^* = 0$, that is, this structure is either dual bipartite or almost dual bipartite.*

As a consequence of Theorems 1, 2 and the classification of distance-regular graphs of diameter at least four that are either dual bipartite or almost dual bipartite [5], we obtain the following result.

Theorem 3. *Let Γ denote a twice Q -polynomial distance-regular graph with diameter d at least 4 and valency at least 3. Then Γ is one of the following:*

- (i) *the cube $H(d, 2)$ with d even;*
- (ii) *the half cube $\frac{1}{2}H(2d+1, 2)$;*
- (iii) *the folded cube $\tilde{H}(2d+1, 2)$;*
- (iv) *the dual polar graph on $[^2A_{2d-1}(q)]$, where $q \geq 2$ is a prime power;*
- (v) *a Hadamard graph of order 2γ with intersection array $\{2\gamma, 2\gamma-1, \gamma, 1; 1, \gamma, 2\gamma-1, 2\gamma\}$ with $\gamma = 1$ or γ a positive even integer.*

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Remark 1. It is known that Hadamard graphs of order 2γ exist for γ any non-negative power of two. A Hadamard graph of order 2γ exists if and only if a Hadamard matrix of 2γ exists [2, Section 1.8]. A Hadamard matrix of order n with n a positive integer is a square $\{+1, -1\}$ matrix H of order n such that $HH^T = nI$. The Hadamard conjecture states that a Hadamard matrix exists if and only if $n = 1, 2$, or n is a positive integer divisible by 4.

Distance-regular graphs of diameter 2 are strongly regular graphs, which possess two P -polynomial and two Q -polynomial structures. Any connected distance regular graph with valency two is an ordinary n -gon, which can have more than two Q -polynomial structures only if $n \geq 7$. So in the rest of this note, we restrict ourselves to distance-regular graphs with both diameter and valency at least three unless stated otherwise.

2. Definitions and preliminaries

In this section, we review some definitions and basic concepts. See the books of Brouwer, Cohen, and Neumaier [2] or Bannai and Ito [1] for more background information.

2.1. Distance-regular graphs

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X , edge set R , path-length distance function ∂ , and diameter $d = \max\{\partial(x, y) \mid x, y \in X\}$. For all $x \in X$ and for all integers i , we set $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$. We abbreviate $\Gamma(x) = \Gamma_1(x)$. The *valency* of a vertex $x \in X$ is the cardinality of $\Gamma(x)$. The graph Γ is said to be *regular*, *valency* k , if each vertex in X has valency k . Graph Γ is said to be *distance-regular* whenever for all integers h, i, j ($0 \leq h, i, j \leq d$), and for all $x, y \in X$ with $\partial(x, y) = h$, the number $p_{i,j}^h = |\Gamma_i(x) \cap \Gamma_j(y)|$ is independent of x and y . When there is no possibility of confusion, we write p_{ij}^h instead $p_{i,j}^h$. The constants p_{ij}^h are known as the *intersection numbers* of Γ .

For notational convenience, set $c_i = p_{1,i-1}^i$ ($1 \leq i \leq d$), $a_i = p_{1i}^i$ ($0 \leq i \leq d$), $b_i = p_{1,i+1}^i$ ($0 \leq i \leq d-1$), $k_i = p_{ii}^0$ ($0 \leq i \leq d$), and define $c_0 = 0$, $b_d = 0$. We note $a_0 = 0$, $k_1 = b_0$, and $c_1 = 1$. We write $k = k_1$. The sequence

$$\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$$

is called the *intersection array* of Γ .

From now on, $\Gamma = (X, R)$ will denote a distance-regular graph of diameter $d \geq 3$. Observe that Γ is regular with valency k , and that

$$k = c_i + a_i + b_i \quad (0 \leq i \leq d). \quad (1)$$

We now recall the Bose-Mesner algebra. For each integer i ($0 \leq i \leq d$), let A_i denote the i -th distance matrix with x, y entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i, \\ 0, & \text{otherwise,} \end{cases} \quad (x, y \in X).$$

Then

$$A_0 = I, \quad A_0 + A_1 + \dots + A_d = J, \quad A_i^t = A_i, \quad A_i A_j = \sum_{k=0}^d p_{ij}^k A_k, \quad (2)$$

where J is the all-one matrix.

We abbreviate $A = A_1$, and refer to this as the *adjacency matrix* of Γ . Let M denote the algebra generated by A over the reals \mathbb{R} . We refer to M as the *Bose-Mesner algebra* of Γ . The matrices A_0, A_1, \dots, A_d form a basis for M . Note that M is closed under the Schur (entry-wise) product \circ . So it has a second basis E_0, E_1, \dots, E_d of primitive idempotents which satisfy

$$E_0 = |X|^{-1}J, \quad E_0 + E_1 + \dots + E_d = I, \quad E_i^t = E_i, \quad E_i \circ E_j = |X|^{-1} \sum_{k=0}^d q_{ij}^k E_k. \quad (3)$$

The numbers q_{ij}^k are nonnegative reals, and are referred to as *Krein parameters*. Note the parameters q_{ij}^k and p_{ij}^k depend on the orderings $(E_i)_i$, and $(A_i)_i$, respectively.

Let $\theta_0, \theta_1, \dots, \theta_d$ denote the real numbers satisfying $A = \sum_{i=0}^d \theta_i E_i$. We refer to θ_i as the *eigenvalue* of Γ associated with E_i , and call θ_0 the *trivial eigenvalue*. If $\theta_0 > \theta_1 > \dots > \theta_d$, then we say E_0, E_1, \dots, E_d is the *natural ordering* of the primitive idempotents.

A *d-class association scheme* is a pair $(X, \{R_0, \dots, R_d\})$ with R_i symmetric binary relations on X whose adjacency matrices satisfy (2), where R_i has *adjacency matrix* A_i defined by $(A_i)_{xy} = 1$ if $(x, y) \in R_i$ and 0 otherwise. It also have a set of primitive idempotents that satisfy (3). The property that one of the relations of a d -class association scheme forms a distance-regular graph with diameter d is equivalent to the scheme being *P-polynomial*, that is, the relations R_0, \dots, R_d can be ordered such that every pair of vertices in R_i has distance i in the graph (X, R_1) for every i . In turn, this is equivalent to the conditions $p_{1i}^{i+1} > 0$ for $0 \leq i \leq d-1$ and $p_{1i}^k = 0$ for $k > i+1$.

2.2. Cosines

We now recall the cosines. Let θ denote an eigenvalue of Γ , and let E denote the associated primitive idempotent. Let $\sigma_0, \sigma_1, \dots, \sigma_d$ denote the real numbers satisfying

$$E = |X|^{-1} m \sum_{i=0}^d \sigma_i A_i, \quad (4)$$

where m denotes the multiplicity of θ . Taking the trace in (4), we find $\sigma_0 = 1$. We call σ_i the i -th cosine of Γ with respect to θ (or E), and call $\sigma_0, \sigma_1, \dots, \sigma_d$ the cosine sequence of Γ associated with θ (or E).

We will need the following basic results.

Lemma 4. [2, Section 4.1.B] *Let Γ denote a distance-regular graph with diameter $d \geq 3$. Then for any complex numbers $\theta, \sigma_0, \sigma_1, \dots, \sigma_d$, the following are equivalent.*

- (i) θ is an eigenvalue of Γ , and $\sigma_0, \sigma_1, \dots, \sigma_d$ is the associated cosine sequence.
- (ii) $\sigma_0 = 1$, and

$$c_i \sigma_{i-1} + a_i \sigma_i + b_i \sigma_{i+1} = \theta \sigma_i \quad (0 \leq i \leq d), \quad (5)$$

where σ_{-1} and σ_{d+1} are indeterminates.

- (iii) $\sigma_0 = 1$, $k\sigma = \theta$, and

$$c_i(\sigma_{i-1} - \sigma_i) - b_i(\sigma_i - \sigma_{i+1}) = k(\sigma - 1)\sigma_i \quad (1 \leq i \leq d), \quad (6)$$

where σ_{d+1} is an indeterminate. □

The second largest and minimal eigenvalue of a distance-regular graph turn out to be of particular interest. In the next several lemmas, we give some basic information on these eigenvalues.

Lemma 5. [8, Lem. 13.2.1] *Let Γ denote a distance-regular graph with diameter $d \geq 3$, and eigenvalues $\theta_0 > \theta_1 > \dots > \theta_d$. Let θ denote one of θ_1, θ_d and let $\sigma_0, \sigma_1, \dots, \sigma_d$ denote the cosine sequence for θ .*

- (i) Suppose $\theta = \theta_1$. Then $\sigma_0 > \sigma_1 > \dots > \sigma_d$.
- (ii) Suppose $\theta = \theta_d$. Then for each i ($0 \leq i \leq d$), $(-1)^i \sigma_i > 0$. □

Lemma 6. *Let Γ be a distance-regular graph with diameter $d \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \dots > \theta_d$. Then (i)–(iii) below hold.*

- (i) $0 < \theta_1 < k$.
- (ii) $a_1 - k \leq \theta_d < -1$.
- (iii) If Γ is not bipartite, then $a_1 - k < \theta_d$. □

2.3. Q -polynomial property

Let $\theta_0, \theta_1, \dots, \theta_d$ (or E_0, E_1, \dots, E_d) be a fixed ordering of the eigenvalues (or primitive idempotents) of Γ . We call this ordering is a Q -polynomial structure if there is a sequence $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_d)$ and polynomial q_j of degree j , $j = 0, 1, \dots, d$, such that

$$E_j = \sum_{i=0}^d q_j(\sigma_i) A_i;$$

in this case, σ is called a Q -sequence of Γ and E_1 is called the primary idempotent for this Q -sequence. The graph Γ is called Q -polynomial if Γ has a Q -polynomial structure.

Let E_0, E_1, \dots, E_d be a Q -polynomial structure for Γ . We usually write $a_i^* = q_{1,i}^i$, $b_i^* = q_{1,i+1}^i$, $c_i^* = q_{1,i-1}^i$ and $k_i^* = q_{ii}^0$ for $i = 0, 1, \dots, d$.

We will need the following theorem of H. Suzuki.

Theorem 7. [18] *Suppose that \mathcal{X} is a symmetric association scheme with a Q -polynomial structure E_0, E_1, \dots, E_d . If \mathcal{X} is not a polygon and has another Q -polynomial structure, then the new structure is one of the following:*

- (I) $E_0, E_2, E_4, E_6, \dots, E_5, E_3, E_1$;
- (II) $E_0, E_d, E_1, E_{d-1}, E_2, E_{d-2}, E_3, E_{d-3}, \dots$;
- (III) $E_0, E_d, E_2, E_{d-2}, E_4, E_{d-4}, \dots, E_{d-5}, E_5, E_{d-3}, E_3, E_{d-1}, E_1$;
- (IV) $E_0, E_{d-1}, E_2, E_{d-3}, E_4, E_{d-5}, \dots, E_5, E_{d-4}, E_3, E_{d-2}, E_1, E_d$;
- (V) $d = 5$ and $E_0, E_5, E_3, E_2, E_4, E_1$.

Hence, \mathcal{X} admits at most two Q -polynomial structures.

Case (V) was recently eliminated in [15].

2.4. Almost dual primitivity

The graph Γ is called *imprimitive* when some i , $1 \leq i \leq d$, the distance- i graph $\Gamma_i = (X, A_i)$ is disconnected. If Γ is imprimitive, then by [2, Theorem 4.2.1], Γ is *bipartite* (here Γ_2 is disconnected) or *antipodal* (here Γ_d is a union of cliques).

A Q -polynomial structure $(E_i)_{i=0}^d$ is called *dual bipartite* if $a_0^* = a_1^* = \dots = a_d^* = 0$. When there is no possibility of confusion, we also say that graph Γ is dual bipartite. Similar comment applies the other concepts to follow immediately. If $c_i^* = b_{d-i}^*$ for $i = 0, 1, \dots, d$ and $i \neq \lfloor d/2 \rfloor$, then Γ is called *dual antipodal*. An imprimitive Q -polynomial distance-regular graph is either dual bipartite or dual antipodal (or both).

Now we define the terms of almost dual bipartite/antipodal, introduced by Dickie [5]. A Q -polynomial structure $(E_i)_{i=0}^d$ is called *almost dual bipartite* if $a_0^* = a_1^* = \dots = a_{d-1}^* = 0 \neq a_d^*$; it is called *almost dual antipodal* if $q_{1d}^d \neq 0 = q_{2d}^d = \dots = q_{dd}^d$. If Γ is almost dual bipartite or antipodal, then it is called *almost dual imprimitive*.

For a classification of almost dual imprimitive distance-regular graphs, see [5, Theorem 3.1.4] for the almost dual bipartite case with $d \geq 4$, and [5, Theorem 2.1.2],[7] for the dual bipartite case with $d \geq 3$.

2.5. The tight property

Now we recall the tight property [9]. A distance-regular graph Γ is called *tight* if it is not bipartite and the following equality holds

$$\left(\theta_1 + \frac{k}{a_1 + 1}\right) \left(\theta_d + \frac{k}{a_1 + 1}\right) = -\frac{ka_1b_1}{(a_1^2 + 1)^2},$$

where θ_1 and θ_d are the second largest and the smallest eigenvalues of Γ , respectively.

3. Proof of Theorem 3

We prove our main theorems in this section. In the rest of this paper, we will fix Ω to be a general distance-regular graph and Γ to be a twice Q -polynomial distance-regular graph of diameter 4. Let E_0, \dots, E_4 and $\tilde{E}_0, \dots, \tilde{E}_4$ be Q -polynomial structures for Γ . The parameters for $(E_i)_i$ will be attached with a tilde.

We first prove Theorem 2, which is key to the proof of Theorem 3.

3.1. Proof of Theorem 2

We first quote some results from Dickie's thesis [5].

Theorem 8. [5, Theorem 7.1.1],[6] Let $\Omega = (X, R)$ denote a distance regular graph with diameter $d \geq 3$. Suppose that Ω admits more than one Q -polynomial structure. Then Ω is thin and dual thin. \square

The follow result follows from Theorem 8 and Theorems 4.1.1 and 5.1.1. in [5].

Theorem 9. [5, Theorem 5.1.2] Let Ω be as in Theorem 8 with a Q -polynomial structure E_0, E_1, \dots, E_d , Krein parameters q_{1i}^i and intersection numbers p_{1i}^i . Then we have the following implications:

$$q_{11}^1 = 0 \quad \Rightarrow \quad q_{1i}^i = 0 \tag{7}$$

$$q_{11}^1 \neq 0 \quad \Rightarrow \quad q_{1i}^i \neq 0 \tag{8}$$

$$p_{11}^1 = 0 \quad \Rightarrow \quad p_{1i}^i = 0 \tag{9}$$

for all $i = 1, 2, \dots, d-1$.

The dual of (8), i.e., $p_{11}^1 \neq 0 \Rightarrow p_{1i}^i \neq 0, 1 \leq i \leq d-1$, holds for any distance-regular graph [2, p.178]. By [5, Corollary 6.2.4], the graph Ω in Theorem 8 is locally strongly regular: if for $x \in X$, $\Omega(x)$ is the set of vertices adjacent to x , the induced graph on $\Omega(x)$ is strongly regular.

If Γ has $q_{11}^1 = 0$ (or $\tilde{q}_{11}^1 = 0$), then, by (7), it is dual bipartite in case $q_{1d}^d = 0$ (or $\tilde{q}_{11}^1 = 0$) and almost dual bipartite otherwise.

The following result applies when $q_{11}^1 \neq 0$ and $\tilde{q}_{11}^1 \neq 0$.

Lemma 10. [5, Lemma 8.2.1] Let Ω be distance-regular graph with diameter $d \geq 4$. Suppose E_0, E_1, \dots, E_d and $\tilde{E}_0, \tilde{E}_1, \dots, \tilde{E}_d$ are Q -polynomial structures for Ω , with Krein parameters q_{ij}^k and \tilde{q}_{ij}^k , respectively. If $q_{11}^1 \neq 0$ and $\tilde{q}_{11}^1 \neq 0$, then $E_1 = \tilde{E}_d$, $E_d = \tilde{E}_1$ and $d = 4$. \square

By Lemma 10, $\tilde{E}_1 = E_4$ and $\tilde{E}_4 = E_1$. By Theorem 7, the Q -polynomial structures for Γ have type III. The following hold by the Q -polynomial property (see also [18, Theorem 2]):

$$q_{14}^4 = 0 = q_{34}^4, \quad q_{24}^4 \neq 0 \neq q_{23}^4.$$

Pascasio [16] showed the following results.

Theorem 11. [16, Theorem 1.3, Lemma 3.2] Let Ω be a distance regular graph with diameter $d \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \dots > \theta_d$. Let E and F be two primitive idempotents other than E_0 .

- (i) Suppose Ω is tight. Then $E \circ F$ is a scalar multiple of a primitive idempotent H of Ω if and only if E, F are a permutation of E_1, E_d . Moreover, the scalar is $\frac{m_E m_F}{|X| m_H}$.
- (ii) Suppose Ω is bipartite. Then $E \circ F$ is a scalar multiple of a primitive of Ω if and only if at least one of E, F is equal to E_d .
- (iii) Suppose Ω is neither bipartite nor tight. Then $E \circ F$ is never a scalar multiple of a primitive of Ω . \square

Theorem 12. [17, Theorem 1.3] Let Ω be a Q -polynomial distance-regular graph with diameter $d \geq 3$, intersection numbers a_i and Krein parameters a_i^* , $0 \leq i \leq d$. The following are equivalent.

- (i) Γ is tight.
- (ii) Γ is not bipartite and $a_d = 0$.
- (iii) Γ is not bipartite and $a_d^* = 0$.

Since $q_{14}^4 = 0$, $E_1 \circ E_4 = |V\Gamma|^{-1} b_3^* E_3$, where $V\Gamma$ is the vertex set of Γ . Theorem 11 says that θ_4 is the eigenvalue associated with E_4 or \bar{E}_4 . Without loss of generality, we assume that it is E_4 . If Γ is tight, then E_1 is associated with θ_1 . By [17, Theorem 1.5], E_0, E_1, E_2, E_3, E_4 is the natural ordering of the primitive idempotents. Now we denote the eigenvalues of Γ by $\theta_0 > \theta_1 > \theta_2 > \theta_3 > \theta_4$.

Now we prove Theorem 2 by showing that one of a_1^*, \bar{a}_1^* for Γ vanishes. We distinguishing whether Γ is bipartite or not.

3.1.1. Γ is bipartite

Assume that Γ is bipartite. So we have $\theta_4 = -k$ and $m_4 = 1$. Imprimitve Q -polynomial association schemes have the following characterization.

Theorem 13. ([19, Theorem 3], [3, 20]) Let E_0, E_1, \dots, E_d be a Q -polynomial structure for association scheme \mathcal{X} . Suppose that \mathcal{X} is imprimitive. More precisely, let T be a proper subset of $\{0, 1, \dots, d\}$ with $T \neq \{0\}$ such that the linear span of $\{E_i \mid i \in T\}$ is closed under the Schur product. In addition, assume $m_1 > 2$. Then one of following holds:

- (i) $T = \{0, 2, 4, \dots\}$ and $a_i^* = 0$.
- (ii) $T = \{0, d\}$ and $b_i^* = c_{d-i}^*$ for all $i = 0, 1, \dots, d$ with the possible exception $i = \lfloor d/2 \rfloor$.

An association scheme \mathcal{X} in case (i) and (ii) is also called *dual bipartite* and *dual antipodal* respectively.

Now back to Γ . Let $m_1 = m_{E_1}$. Suppose $m_1 > 2$. Then Theorem 13 applies. If $q_{11}^1 > 0$, Γ can not be dual bipartite and hence Γ is dual antipodal, i.e., case (ii). So $T = \{0, 4\}$. However, E_4 is the primary idempotent for the second Q -polynomial structure, which is impossible.

Suppose $m_1 \leq 2$. Since $m_{E_1} < k = 3$, we have $\theta_{E_1} = \theta_1$ by [2, Theorem 4.4.4]. If $m_1 = 2$, then by [14, Theorem 13 (i)] $k = 2$, this contradicts $k > 2$. By [14, Lemma 7], it is impossible for $m_1 = 1$; otherwise $m_1 + m_4 = 2 < k$.

3.1.2. Γ is not bipartite

Assume that Γ is not bipartite. Since $a_4^* = 0$, Γ is tight and $a_4 = 0$ by Theorem 12. In the literature [2, p.247], there is an infinite series of feasible formally self-dual intersection arrays

$$\{\mu(2\mu+1), (\mu-1)(2\mu+1), \mu^2, \mu; 1, \mu, \mu(\mu-1), \mu(2\mu+1)\} \quad (10)$$

This series was ruled out in [11]. Had a graph with this array existed, it would be tight with a pair of non-integral eigenvalues, and would have possessed two P -polynomial and two Q -polynomial structures. We will show below that there are no distance-regular graphs with intersection array (10) in an alternative way.

Since Γ is not bipartite, it follows from (8) that $a_1 a_2 a_3 \neq 0$, $a_4 = 0$. We can deduce from Theorem 11 (i) and Lemma 4 (iii) that

$$\theta_1 \theta_4 = \theta_0 \theta_3. \quad (11)$$

Theorem 14. [2, Theorem 8.1.2, Corollary 8.1.4] Let Ω be a Q -polynomial distance-regular graph. Then every Q -sequence $(\sigma_0, \sigma_1, \dots, \sigma_d)$ of Ω satisfies the recurrence

$$\sigma_{i+1} + \sigma_{i-1} = p\sigma_i + r \quad (i = 1, \dots, d-1), \quad (12)$$

for suitable numbers p and r .

If $k = \theta_0, \theta_1, \dots, \theta_d$ the Q -polynomial structure corresponding to the above Q -sequence, then there are constants r^*, s^* such that

$$\left. \begin{aligned} \theta_{\ell+1} + \theta_{\ell-1} &= p\theta_\ell + r^* \\ \theta_{\ell+1}\theta_{\ell-1} &= \theta_\ell^2 - r^*\theta_\ell - s^* \end{aligned} \right\} \quad (\ell = 1, \dots, d-1). \quad (13)$$

Now applying Theorem 14 to the two Q -polynomial structures of Γ , we obtain:

$$\theta_2 - p\theta_1 + \theta_0 = \theta_3 - p\theta_2 + \theta_1 = \theta_4 - p\theta_3 + \theta_2, \quad (14)$$

$$\theta_2 - \tilde{p}\theta_4 + \theta_0 = \theta_3 - \tilde{p}\theta_2 + \theta_4 = \theta_1 - \tilde{p}\theta_3 + \theta_2. \quad (15)$$

We obtain from (14) and (15)

$$p = \frac{\theta_0 - \theta_4}{\theta_1 - \theta_3}, \quad \tilde{p} = \frac{\theta_0 - \theta_1}{\theta_4 - \theta_3}, \quad p - \tilde{p} = \frac{2(\theta_1 - \theta_4)}{\theta_2 - \theta_3}.$$

From these equation and $\theta_1\theta_4 = k\theta_3$ we find

$$p = \theta_0/\theta_1, \quad \tilde{p} = \theta_0/\theta_4, \quad \theta_2 = -\theta_3. \quad (16)$$

Now substituting these into (14) leads to

$$\theta_1 + \theta_4 = 2\theta_2. \quad (17)$$

If we substitute (16) into [2, Corollary 8.1.4 (24)], we find

$$r^* = \theta_2, \quad s^* = \theta_1\theta_3, \quad \tilde{r}^* = \theta_2, \quad \tilde{s}^* = \theta_4\theta_2.$$

Since Γ is tight, we have

$$\left(\theta_1 + \frac{k}{a_1 + 1}\right) \left(\theta_4 + \frac{k}{a_1 + 1}\right) = -\frac{ka_1b_1}{(a_1 + 1)^2}. \quad (18)$$

We find from this and Eq. (17) and (11) that

$$\theta_2(a_1 - 1) = b_1 + 1. \quad (19)$$

Now we collect some equations above that are key to the proof to follow:

$$\theta_1\theta_4 = \theta_0\theta_3 \quad (20)$$

$$\theta_1 + \theta_4 = 2\theta_2 \quad (21)$$

$$\theta_2(a_1 - 1) = b_1 + 1. \quad (22)$$

Since Γ is tight, a local graph $\Gamma(x)$ is strongly regular with k vertices and valency a_1 and non-trivial eigenvalues

$$\xi = -1 - \frac{b_1}{(\theta_4 + 1)}, \quad \tau = -1 - \frac{b_1}{(\theta_1 + 1)}.$$

where $\xi \geq 0$ and $\tau < -1$.

The local graph $\Gamma(x)$ can not be a conference graph. Otherwise, we have $a_1 = (k - 1)/2$ and such a graph has diameter 3 by [13]. (The intersection array (10) has the second largest and minimal eigenvalues non-integral, and thus any graph with this array has a conference graph as its local graph and therefore can not exist.) Therefore, σ and τ are both integers and thus θ_4, θ_1 are both rational numbers. Since they are algebraic integers, θ_4, θ_1 are integers.

We find from (20)-(22) that

$$-(\theta_1 + 1)(\theta_4 + 1) = -\theta_1\theta_4 - \theta_1 - \theta_2 - 1 = (k - 2)\theta_2 - 1 = (a_1 - 1)\theta_2 + b_1\theta_2 - 1 = b_1 + 1 + b_1\theta_2 - 1 = b_1(\theta_2 + 1).$$

From this we can derive

$$\frac{-b_1^2}{(\theta_1 + 1)(\theta_4 + 1)} = \frac{b_1}{(\theta_2 + 1)}. \quad (23)$$

Since the left hand side is an integer, $\theta_2 + 1$ divides b_1 and hence a_1 by (22).

Let $(\sigma_i)_i$ be the cosine sequence of θ_1 . Then $\sigma_i > \sigma_{i+1}$ ($0 \leq i \leq 3$) and $\sigma_3 = \sigma_1\sigma_4$. As the sequence σ_i has one sign change, $\sigma_4 < 0$ and $\sigma_1 > 0$. Hence $\sigma_3 < 0$. Let $(\tilde{\sigma}_i)_i$ be cosine sequence of θ_4 . Then $(-1)^i \tilde{\sigma}_i > 0$. Let $(u_i)_i$ be the cosine sequence of θ_3 . Then $u_i = \sigma_i \tilde{\sigma}_i$.

Let $a_1 = \alpha(\theta_2 + 1)$, $b_1 = \beta(\theta_2 + 1)$. By (20) and (21) we find $\theta_1, \theta_4 = \theta_2 \pm \sqrt{\theta_2(k + \theta_2)}$. Now $\theta_2(k + \theta_2) = \alpha\theta_2(\theta_2 + 1)^2$ and hence $k = \alpha(\theta_2 + 1)^2 - \theta_2$.

By [4, Proposition 1 (iii), Proposition 2 (iii)], $\sigma_2 \geq 0$ and thus $u_2 \geq 0$. We find from this and $\theta_2 = -\theta_3$ that $k - \theta_2 a_1 \leq \theta_2^2$. As $k = (a_1 - 1)(\theta_2 + 1) + 1$, we obtain $a_1 \leq \theta_2(\theta_2 + 1)$ and thus $\alpha \leq \theta_2$.

Recall $(\alpha(\theta_2 + 1) - 1)\theta_2 - 1 = (a_1 - 1)\theta_2 - 1 = b_1 = \beta(\theta_2 + 1)$. We obtain: $\beta = \alpha\theta_2 - 1$ and $b_1 = (\alpha\theta_2 - 1)(\theta_2 + 1)$, $\xi = -1 - b_1/(\theta_4 + 1) = \sqrt{\alpha\theta_2}$ and $\tau = -\xi$. This means that ξ divides $a_1 (= k - b_1 - 1)$ and hence, as $a_1 = \xi^2 + \alpha$,

ξ divides α . This implies that θ_2 divides α and hence $\theta_2 \leq \alpha$. We conclude $\alpha = \theta_2 = \xi$. Now, the local graph $\Gamma(x)$ has the following parameters:

$$k = \theta_2^2(\theta_2 + 2), \quad a_1 = \theta_2(\theta_2 + 1), \quad \theta_1 = \theta_2(\theta_2 + 2), \quad \theta_4 = -\theta_2^2.$$

We see that

$$\theta_1 = \frac{a_1 + \sqrt{a_1^2 + 4k}}{2}.$$

Now Γ is antipodal by the following result.

Lemma 15. [12, Proposition 3.5] *Let Ω be a distance-regular graph with d at least three and distinct eigenvalues $k = \theta_0 > \theta_1 > \dots > \theta_d$. Then $\theta_1 = (a_1 + \sqrt{a_1^2 + 4k})/2$ if and only if one of the following holds:*

- (i) $d = 3$ and Ω is a Shilla distance-regular graph;
- (ii) $d = 4$ and Ω is an antipodal distance-regular graph.

Since Γ is antipodal, it is dual bipartite. So $a_1^* = 0$, or $\tilde{a}_1^* = 0$. This completes the proof of Theorem 2.

3.2. Proof of Theorem 3

By Lemma 9 and the succeeding remark, the parameters a_i^* of Γ can be divided into the following four cases:

- (i) If $a_1^* = a_4^* = 0$, then Γ is $H(4, 2)$ or a Hadamard graph by [7].
- (ii) If $a_1^* = 0 \neq a_4^*$, then Γ is $\frac{1}{2}H(9, 2)$, or $\tilde{H}(9, 2)$ by [5, Theorem 3.1.4].
- (iii) Suppose $a_1^* \neq 0 \neq a_4^*$. By [18, Theorem 2], the Q -polynomial structures $(E_i)_i$ is almost dual antipodal and thus by [5, Lemma 3.1.3], the other Q -polynomial structure $(\tilde{E}_i)_i$ is almost dual bipartite. This case is implied by the previous case by treating $(\tilde{E}_i)_i$, which has $\tilde{a}_1^* = 0 \neq \tilde{a}_4^*$.
- (iv) If $a_1^* \neq 0 = a_4^*$, then $\tilde{a}_1^* = 0$ by Theorem 2 and Cases (i),(ii) apply.

So the proof of Theorem 3 is completed.

Remark 2. The following graphs are twice Q -polynomial distance-regular graphs of diameter 3: each graph in Theorem 1 (ii)-(iv) with $d = 3$, and a distance-regular graph intersection array $\{k, k - a_1 - 1, 1; 1, k - a_1 - 1, k\}$. If $a_1 = 0$, this array is uniquely realized by the complement of $K_{k+1} \times K_2$; any distance-regular graph with this intersection array is called a *Taylor graph* if $a_1 > 0$; see [2, p.13].

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